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# Unitary transformation approach to the exact solution for the singular oscillator

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**Abstract.** By performing unitary transformations, the exact solution of the time-dependent singular oscillator is obtained. The invariant operator and the auxiliary equation are rigorously established. The non-adiabatic Berry's phase is calculated.

It is well known that an exact and analytical solution of the Schrödinger equation can only be found for a limited number of potentials. If the potentials are, in addition, time-dependent it is very rare to be able to find an exact solution. In a series of papers, an important class of Hamiltonians called the generalized harmonic oscillators was intensively discussed [1–7]. Recently, Fu-li Li *et al* [7], using three consecutive unitary transformations on the time-dependent Schrödinger equation, discussed the exact solution and the invariant operator of a generalized harmonic oscillator. They have shown that the auxiliary functions for constructing the invariant operator are just solutions of a linear differential equation for the associated classical harmonic oscillator with time-dependent mass and frequency.

In the present paper, we apply the same approach as in [7] to the so-called 'singular oscillator', i.e. a quantum particle moving in the potential $V(q,t) = \frac{1}{2}[X(t)q^2 + Z(t)l^2/q^2]$ with time-dependent coefficient. As was shown in [8], the solvability of the singular oscillator problem is explained by the fact that its Hamiltonian is a linear combination of generators of the $SU(1,1)$ algebra.

The one-dimensional system we study is described by the following Hamiltonian:

$$H = \frac{1}{2}\left[Z(t)p^2 + Y(t)(pq+qp) + X(t)q^2 + \frac{Z(t)l^2}{q^2}\right]. \tag{1}$$

It is a linear combination of generators of the algebra $SU(1,1)$ provided that the ratio of the kinetic and nonlinear terms is a constant. In equation (1) $q$ and $p$ are quantum mechanical operators, $X(t)$, $Y(t)$, and $Z(t)$ are arbitrary functions of time, and $l$ is an arbitrary constant which could be zero. We show that the exact solution of the singular oscillator can be found by introducing three consecutive unitary transformations. In contrast to the approach of Fu-li Li *et al* [7], a time-dependent invariant operator for the system and an auxiliary equation appears automatically in this process by setting the global time-dependent frequency for the singular oscillator equal to a real constant. On the basis of the exact solution, a non-adiabatic Berry's phase is calculated.

Now, we consider the singular oscillator with the Hamiltonian given by equation (1). The system evolves in time according to the Schrödinger equation (assume $\hbar = 1$)

$$\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{2}$$

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Suppose that $U(t)$ is a time-dependent transformation such that

$$|\psi_1(t)\rangle = U^{-1}(t)|\psi(t)\rangle. \tag{3}$$

Substituting (3) into (2), we find the equation of motion for $|\psi_1(t)\rangle$,

$$\mathrm{i}\frac{\partial}{\partial t}|\psi_1(t)\rangle = \left[U^{-1}H(t)U - \mathrm{i}U^{-1}\frac{\partial U}{\partial t}\right]|\psi_1(t)\rangle. \tag{4}$$

In the above equation, the new Hamiltonian operator

$$H_1 = U^{-1}H(t)U - \mathrm{i}U^{-1}\frac{\partial U}{\partial t} \tag{5}$$

is Hermitian. This requires that $U(t)$ must be an unitary operator. Since the Hamiltonian (1) of a system is time-dependent, $U(t)$ results in an unitary transformation (3) for the wavefunction and an unitary transformation (5) for the Hamiltonian so that the form of the Schrödinger equation remains form invariant.

The key point of our analysis is to perform the three consecutive transformations

$$|\psi\rangle = U_1|\psi_1\rangle = \exp\left[-\frac{\mathrm{i}}{2}\int_0^t Y(t')\,\mathrm{d}t'(pq+qp)\right]|\psi_1\rangle \tag{6}$$

$$|\psi_1\rangle = U_2|\psi_2\rangle = \exp[\mathrm{i}C_1(t)q^2]|\psi_2\rangle \tag{7}$$

and

$$|\psi_2\rangle = U_3|\psi_3\rangle = \exp[\mathrm{i}C_2(t)(pq+qp)]|\psi_3\rangle \tag{8}$$

where $C_1(t)$ and $C_2(t)$ are real functions of time. These coefficients are chosen in such a way that the Hamiltonian, after these transformations becomes a product of two factors, namely a simple time-independent *singular* oscillator Hamiltonian and a time-dependent factor.

It can easily be shown that under these transformations the coordinate and momentum operators change according to

$$U_1^{-1}pU_1 = p\exp\left[-\int_0^t Y(t')\,\mathrm{d}t'\right] \tag{9}$$

$$U_1^{-1}qU_1 = q\exp\left[\int_0^t Y(t')\,\mathrm{d}t'\right] \tag{10}$$

$$U_1^{-1}\frac{1}{q}U_1 = \frac{1}{q}\exp\left[-\int_0^t Y(t')\,\mathrm{d}t'\right] \tag{11}$$

$$U_2^{-1}pU_2 = p + 2C_1q \tag{12}$$

$$U_2^{-1}qU_2 = q. \tag{13}$$

In order to solve (2) with the Hamiltonian specified by (1) we first try to remove the mixed terms in $q$ and $p$ in the Hamiltonian (1). This can be achieved by the transformation $U_1(t)$. Inserting (6) and (9)–(11) into (4), we have

$$\mathrm{i}\frac{\partial}{\partial t}|\psi_1(t)\rangle = \frac{1}{2}\left[\frac{1}{m(t)}p^2 + k(t)q^2 + \frac{l^2}{m(t)}\frac{1}{q^2}\right]|\psi_1(t)\rangle \tag{14}$$

where

$$m(t) = \frac{1}{Z(t)}\exp\left[2\int_0^t Y(t')\,\mathrm{d}t'\right] \tag{15}$$

$$k(t) = X(t)\exp\left[2\int_0^t Y(t')\,\mathrm{d}t'\right]. \tag{16}$$

Now let us perform the transformation $U_2(t)$. Substituting (7) into (14) and using (12) and (13), we find the equation of motion for $|\psi_2(t)\rangle$:

$$\mathrm{i}\frac{\partial}{\partial t}|\psi_2(t)\rangle = \frac{1}{2}\left[\frac{1}{m}p^2 + 2\frac{C_1}{m}(pq+qp) + 2\left(\dot{C}_1 + 2\frac{C_1^2}{m} + \frac{k}{2}\right)q^2 + \frac{l^2}{m}\frac{1}{q^2}\right]|\psi_2(t)\rangle. \quad (17)$$

In order to remove the cross term in (17), we consider the transformation $U_3(t)$. Since this transformation is formally the same as $U_1(t)$, this suggests that one takes

$$C_2(t) = -\int_0^t \frac{C_1(t')}{m(t')}\,\mathrm{d}t'. \quad (18)$$

With this choice, (17) is changed under the transformation into

$$\mathrm{i}\frac{\partial}{\partial t}|\psi_3(t)\rangle = \frac{1}{2}\left\{\frac{1}{m}\exp\left[-4\int_0^t \frac{C_1}{m}\,\mathrm{d}t'\right]\left(p^2 + \frac{l^2}{q^2}\right) + 2\left(\dot{C}_1 + 2\frac{C_1^2}{m} + \frac{k}{2}\right)\right.$$
$$\left. \times \exp\left[4\int_0^t \frac{C_1}{m}\,\mathrm{d}t'\right]q^2\right\}|\psi_3(t)\rangle. \quad (19)$$

Choosing

$$C_1(t) = \frac{\sqrt{W}m}{2}\left(\frac{\dot{\rho}}{\rho} - Y\right) \quad (20)$$

where $W$ is a real constant and $\rho(t)$ is a real function of time, one can show that (19) becomes

$$\mathrm{i}\frac{\partial}{\partial t}|\psi_3(t)\rangle = \frac{1}{2}\frac{Z}{W\rho^2}\left[p^2 + \Omega^2 q^2 + \frac{l^2}{q^2}\right]|\psi_3(t)\rangle \quad (21)$$

where

$$\Omega^2 = W^2\frac{\rho^3}{Z^2}\left(\ddot{\rho} - \frac{\dot{Z}}{Z}\dot{\rho} + \left[(XZ - Y^2) + \frac{\dot{Z}}{Z}Y - \dot{Y}\right]\rho\right). \quad (22)$$

The central idea in this procedure is to require that the Hamiltonian governing the evolution of $|\psi_3\rangle$ is a product of two parts: a simple time-independent singular oscillator and a time-dependent factor. Let us set the global time-dependent frequency appearing in (21) equal to a real constant

$$\Omega^2 = W^2 \quad (23)$$

which amounts to imposing a constraint on $\rho$. Equation(22) then becomes

$$\ddot{\rho} - \frac{\dot{Z}}{Z}\dot{\rho} + \left[(XZ - Y^2) + \frac{\dot{Z}}{Z}Y - \dot{Y}\right]\rho = \frac{Z^2}{\rho^3} \quad (24)$$

which is the well known auxiliary equation [8].

In order to obtain the invariant operator, consider the operator

$$I(t) = U_1U_2U_3\frac{1}{2}\left(p^2 + W^2q^2 + \frac{l^2}{q^2}\right)U_3^{-1}U_2^{-1}U_1^{-1}$$
$$= \frac{W}{2}\left[\rho^2 + p^2 + \left(\frac{\rho^2 Y - \rho\dot{\rho}}{Z}\right)(pq+qp) + \frac{1}{\rho^2}\left\{1 + \left(\frac{\rho^2 Y - \rho\dot{\rho}}{Z}\right)^2\right\}q^2 + \left(\frac{\rho l}{q}\right)^2\right]. \quad (25)$$

Thus $I(t)$ is exactly the invariant operator [8], which satisfies the equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \mathrm{i}[H, I] = 0. \quad (26)$$

Obviously, one can write the solution of (21) as

$$|\psi_3(t)\rangle = \exp\left[-\mathrm{i}\int_0^t \frac{Z}{W\rho^2}\,\mathrm{d}t'\frac{1}{2}\left(p^2+W^2q^2+\frac{l^2}{q^2}\right)\right]|\psi_3(0)\rangle. \tag{27}$$

Note that the evolution of $|\psi_3\rangle$ is governed by a Hamiltonian which can be interpreted as the radial part of a three-dimensional harmonic oscillator, where $l^2 = c(c+1)$ is related to the angular momentum. To get an explicit form of the solution, let us assume the initial state

$$|\psi_3(0)\rangle = |n, c\rangle \tag{28}$$

which satisfies the eigenequation

$$\frac{1}{2}\left(p^2+W^2q^2+\frac{l^2}{q^2}\right)|n,c\rangle = 2\left(n+\frac{c}{2}+\frac{3}{4}\right)W|n,c\rangle \qquad n = 0, 1, 2, \ldots. \tag{29}$$

Inserting (29) into (27), we have

$$|\psi_3(t)\rangle = \exp\left[-2\mathrm{i}\left(n+\frac{c}{2}+\frac{3}{4}\right)\int_0^t \frac{Z}{\rho^2}\,\mathrm{d}t'\right]|n,c\rangle. \tag{30}$$

The exact solution of the original equation (2), can now be found by combining the above results. We finally obtain

$$|\psi_n^s(t)\rangle = \exp\left[-2\mathrm{i}\left(n+\frac{c}{2}+\frac{3}{4}\right)\int_0^t \frac{Z}{\rho^2}\,\mathrm{d}t'\right]U_1U_2U_3|n,c\rangle. \tag{31}$$

In the $q$-representation, the state vector (31) takes the form

$$\psi_n^s(q,t) = \exp\left[-2\mathrm{i}\left(n+\frac{c}{2}+\frac{3}{4}\right)\int_0^t \frac{Z}{\rho^2}\,\mathrm{d}t'\right]\psi_n(q,t) \tag{32}$$

where

$$\begin{aligned}\psi_n(q,t) = {}&\left[\frac{2}{\rho}\frac{\Gamma(n+1)}{\Gamma(n+c+3/2)}\right]^{1/2}\exp\left[-\frac{1}{2}\left(\frac{q}{\rho}\right)^2\left\{1+\mathrm{i}\frac{Y\rho^2-\dot{\rho}\rho}{Z}\right\}\right]\\ &\times\left(\frac{q}{\rho}\right)^{(c+1)/4}L_n^{c+1/2}\left(\frac{q}{\rho}\right)\end{aligned} \tag{33}$$

and $L_n^{c+1/2}$ denotes the generalized Laguerre polynomials. Applying $I(t)$ to (33) we can show that $\psi_n(q,t)$ is an eigenstate of $I(t)$ with eigenvalue $2(n+c/2+3/4)W$. Since $\psi_n^s(q,t)$, the solution of (2) with initial condition (28) differs from $\psi_n(q,t)$ only by the time-dependent phase

$$\alpha_n(t) = -2\left(n+\frac{c}{2}+\frac{3}{4}\right)\int_0^t \frac{Z}{\rho^2}\,\mathrm{d}t' \tag{34}$$

the general solution of (2) can be written in the form

$$\psi(q,t) = \sum_n a_n \exp\left[-2\mathrm{i}\left(n+\frac{c}{2}+\frac{3}{4}\right)\int_0^t \frac{Z(t')}{\rho^2}\,\mathrm{d}t'\right]\psi_n(q,t) \tag{35}$$

where $a_n$ are constants to be determined by the initial condition $\psi_n(q,0)$. Expression (35) is just what is anticipated by the Lewis–Riesenfeld theory [9].

Now let us calculate the Berry's phase [10] for the singular oscillator from the state vector in the $q$-representation $\psi_n^s(q,t)$ (32). Suppose that at $t=0$, the system is in the $n$th eigenstate of the invariant operator $I(t)$. It will evolve into the state specified by (32) at a later time $t$. If the parameters $X(t)$, $Y(t)$, and $Z(t)$ are periodic functions of time with period $T$, i.e. $(X, Y, Z)(t+T) = (X, Y, Z)(t)$, equation (24) may have periodic solutions. When the auxiliary function is periodic, i.e. $\rho(t+T) = \rho(t)$, then after one period of evolution the system returns to the initial state except for acquiring the total phase

$$\alpha_n(T) = -2\left(n + \frac{c}{2} + \frac{3}{4}\right)\int_0^T \frac{Z}{\rho^2}\,\mathrm{d}t. \tag{36}$$

The conventional dynamic phase obtained over one period is

$$\alpha_n^D(T) = -\int_0^T \langle\psi_n|H|\psi_n\rangle\,\mathrm{d}t = -2\left(n + \frac{c}{2} + \frac{3}{4}\right)\int_0^T \left\{\frac{Z}{\rho^2} - \left[\frac{Y\rho^2 - \dot{\rho}\rho}{Z}\right]\frac{\dot{\rho}}{\rho}\right\}\mathrm{d}t. \tag{37}$$

Therefore, the non-adiabatic Berry's phase in a cyclic evolution [11] is given by

$$\alpha_n^B(\mathcal{C}) = \alpha_n - \alpha_n^D = -2\left(n + \frac{c}{2} + \frac{3}{4}\right)\oint_{\mathcal{C}}\left[\frac{Y\rho - \dot{\rho}}{Z}\right]\mathrm{d}\rho \tag{38}$$

where $\mathcal{C}$ is a closed circuit in the parameter space.

In conclusion, we have found the exact solution of the time-dependent Schrödinger equation for the singular oscillator by performing unitary transformations. The invariant operator and the auxiliary equation are obtained. The calculation does not presuppose the existence of an invariant operator or the knowledge of an auxiliary equation. We have also found the exact expression of the non-adiabatic Berry's phase.

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